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FUNDAMENTAL SOLUTION OF A NET HEAT CONDUCTION EQUATION

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Computational procedures using a discrete analog of Green's function are introduced and substantiated by the example of a solution to a net linear steady-state equation of heat conduction.

Using a discrete analog of Green's function enables us to pass to finite sums rather than to a system of difference (algebraic) equations. In doing so for a linear problem with constant coefficients we can obtain a general solution for the specified geometry and the adopted net, that does not depend on the form or values of the boundary conditions or the value or coordinates of heat-release sources (similar solutions are partially obtained in [1] by another method).

As in the continuous case, a fundamental solution obtained using a discrete analog of Green's function is a solution to the simplest problem: the distribution of the temperature field for specified unit heat releases and zero boundary conditions. Then using the fundamental solution we find solutions to more complicated problems with boundary conditions that are more complex (but linear) and arbitrary in form and value and, with an arbitrary value and geometry of heat releases that are constant in time (this involves linear problems with constant coefficients).

We describe computational procedures first, and then we substantiate them. Let it be necessary to determine an l-dimensional ($l = 1, 2, 3$) steady-state temperature field at the nodes of a net of a body, in which the process of temperature distribution is described by an l-dimensional linear equation of steady-state heat conduction with constant coefficients. In accordance with the general rules [2, 3] we replace the region of continuous temperature variation by a uniform net with nodes of steps h and r (an example of a two-dimensional net is given in Fig. 1). The nodes of the net are numbered in the following order: $k = 1, 2, \ldots, r-s$ are boundary nodes, $i, j = r-s+1, \ldots, r$ are internal nodes. The matrix of the initial data of size $s \times r$ has rows with numbers corresponding to the internal nodes $i, j = r-s+1, \ldots, r$ and columns with numbers of all the nodes $d = 1, 2, \ldots, r$. The rows of the initial data matrix are filled by the elements a_{id} according to the rule

$$a_{id} = \left| \begin{array}{ll} (2l)^{-1} & \text{for nodes adjacent to } i\,; \\ 0 & \text{for the remaining nodes}\,. \end{array} \right. \tag{1}$$

The matrix of the initial data consists of two submatrices: R of size $s \times (r-s)$ and Q of size $s \times s$; each row has $2l$ nonzero elements; the sum of the elements in each row is equal to 1 (a stochastic matrix). Figures 1 and 2 give the numbering of the nodes and the filling in of the initial data matrix for our example.

A fundamental solution of a net linear steady-state problem of heat conduction is determined from the expression

$$N = (E - Q)^{-1} = \sum_{m=0}^{\infty} Q^m = E + Q + Q^2 + Q^3 + \ldots \tag{2}$$

Since $\lim\limits_{m \to \infty} Q^m = 0$ (we will show this below), the number of terms in the sum in (2) can be limited by the value $m = m_1$ such that the elements of the matrix Q^{m_1} are smaller in absolute value (in fractions of unity) than the error adopted by us for this calculation.

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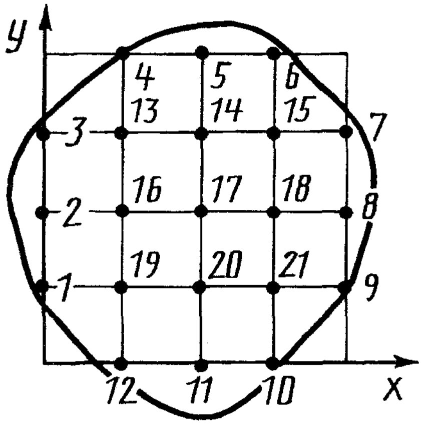

i	d																				
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
13			+	+										+		+					
14					+					R			+		+		+			Q	
15						+	+							+				+			
16		+											+				+		+		
17														+		+		+		+	
18								+							+		+				+
19	+											+				+				+	
20											+						+		+		+
21									+	+								+		+	

Fig. 1. Two-dimensional net for the example: 1-12) boundary nodes, 13-21) internal nodes.

Fig. 2. Submatrices and matrices of transition probabilities. The sign + corresponds to 0.25, empty cells correspond to 0.

The elements n_{ij} of matrix (2) (here $i, j = r-s+1, \ldots, r$ correspond to the number of an internal node) show the value of the increment of the steady-state temperature at node j on condition that heat of power $q_{v1} = 2l\lambda/h^2$ is constantly released at node i under zero boundary conditions ($n_{ij} = n_{ji}$).

Thus, the matrix N of (2) is a fundamental solution to the linear net equation of steady-state heat conduction with zero boundary conditions, and the elements n_{ij} of this matrix are a discrete analog of Green's function.

From the fundamental solution we can determine the coefficients of influence (a discrete analog of Green's formula):

$$B = NR\,. \tag{3}$$

Matrix (3) has the dimensions $s \times (r-s)$ and the elements b_{ik} of the matrix (here $i = r-s+1, \ldots, r$ is an internal node number, $k = 1, 2, \ldots, r-s$ is a boundary node number) show the influence of the temperature of boundary node k on the temperature of internal node i.

From (2) and (3) the solution of the linear problem of steady-state heat conduction with boundary conditions of the first kind specified by the temperatures T_k at boundary nodes $k = 1, 2, \ldots, r-s$ and with constant heat releases whose value and coordinates are specified by the values of q_{vj} (W/m^3) is determined by the values of the s temperatures T_i at the internal nodes

$$T_i = \sum_{k=1}^{r-s} b_{ik}\, T_k + \sum_{j=r-s+1}^{r} n_{ij} \frac{q_{vj}\, h^2}{2l\lambda}\,. \tag{4}$$

When specifying boundary conditions of the II and III kind we must use a possible approximation of derivatives by finite differences, for example:

II

$$T_k = \frac{q_s\, h}{\lambda} + T_i\,;$$

III

$$T_k = \frac{\lambda}{\lambda + \alpha_k h} T_i + \frac{\alpha_k h}{\lambda + \alpha_k h} T_c\,, \tag{5}$$

TABLE 1. For the Solution of the Example - the Matrices N and B

	13	14	15	16	17	18	19	20	21
13	*a*	*b*	*c*	*b*	*d*	*e*	*c*	*e*	*f*
14	*b*	*g*	*b*	*d*	*h*	*d*	*e*	*i*	*e*
15	*c*	*b*	*a*	*e*	*d*	*b*	*f*	*e*	*c*
16	*b*	*d*	*e*	*g*	*h*	*i*	*b*	*d*	*e*
17	*d*	*h*	*d*	*h*	*j*	*h*	*d*	*h*	*d*
18	*e*	*d*	*b*	*i*	*h*	*g*	*e*	*d*	*b*
19	*c*	*e*	*f*	*b*	*d*	*e*	*a*	*b*	*c*
20	*e*	*i*	*e*	*d*	*h*	*d*	*b*	*g*	*b*
21	*f*	*e*	*c*	*e*	*d*	*b*	*c*	*b*	*a*

$\times 10^{-6} = N$

	1, 12	2	3, 4	5	6, 7	8	9, 10	11
13	*k*	*n*	*p*	*n*	*k*	*l*	*m*	*l*
14	*l*	*o*	*n*	*q*	*n*	*o*	*l*	*s*
15	*m*	*l*	*k*	*n*	*p*	*n*	*k*	*l*
16	*n*	*q*	*n*	*o*	*l*	*s*	*l*	*o*
17	*o*	*r*	*o*	*r*	*o*	*r*	*o*	*r*
18	*l*	*s*	*l*	*o*	*n*	*q*	*n*	*o*
19	*p*	*n*	*k*	*l*	*m*	*l*	*k*	*n*
20	*n*	*o*	*l*	*s*	*l*	*o*	*n*	*q*
21	*k*	*l*	*m*	*l*	*k*	*n*	*p*	*n*

$\times 10^{-6} = B$

Note. In the matrix B columns 1 and 12, 3 and 4, 6 and 7, 9 and 10 are the same; *a*) 1196300, *b*) 392613, *c*) 124878, *d*) 249756, *e*) 106899, *f*) 53449, *g*) 1321190, *h*) 488512, *i*) 178327, *j*) 1499510, *k*) 31219, *l*) 26725, *m*) 13362, *n*) 98153, *o*) 62439, *p*) 299075, *q*) 330300, *r*) 124876, *s*) 44582.

where T_i is the internal node adjacent to T_k in a direction close to the normal to the surface (for bodies of intricate form the net should be fairly fine so that the choice of the node T_i will not lead to a large error). The value of (5) should be substituted for the temperature T_k of those nodes in (4) in which boundary conditions of the II and (or) the III kind are specified. For our example (Figs. 1 and 2) the matrices of a fundamental solution from (2) and (3) are given in Table 1. We find the solution after substituting values from the table into (4); for example, by substituting the initial data of the example from [3, pp. 229-232] we obtain the same results.

We substantiate the computational procedure given above. The association of the heat conduction equation with a Markov process was established as early as the 19–20s [4]. A discrete approximation of a Markov process is the process of a random walk over the nodes of a net [5] (convergence is proved in [6]). The random walk method is used to solve the heat conduction problem by the Monte Carlo method [5, 7, 8]. When approximating the heat conduction equation by finite differences the net is constructed as described above. A difference analog that approximates the initial boundary-value problem can be constructed by various methods; widest acceptance has been received by the method of formal replacement of derivatives by finite differences that contain the values of net functions at the nodes of the net. Thus, for our example (Fig. 1) we obtain s equations (according to the number of internal nodes) [2, 3, 5]:

$$T(x, y) = \frac{1}{4}\left[T(x+h, y) + T(x-h, y) + T(x, y+h) + T(x, y-h) + \frac{q_v(x, y)\, h^2}{\lambda}\right].$$

In a random walk over the nodes of the net (Fig. 1) this equation is interpreted in the following way: $T(x, y)$ goes over to adjacent nodes with the same probability $p = 1/4$ until it reaches a boundary node, where it is absorbed. If there is a heat source at the internal node (x, y), the value of the temperature increases [7, 8]. Random walks of this type are a Markov absorbing chain, whose solution makes it possible to determine a steady-state temperature field. The transition probability matrix has the form [9]

$$P = \begin{array}{c} \\ \\ \end{array}\begin{array}{|c|c|} \multicolumn{1}{c}{\overbrace{\quad}^{r-s}} & \multicolumn{1}{c}{\overbrace{\quad}^{s}} \\ E & 0 \\ \hline R & Q \end{array} \begin{array}{l} \\ \left.\right\} \; r-s \\ \left.\right\} \; s \end{array}.$$

(The matrices R and Q for our example are given in Fig. 2.) Here the zero submatrix corresponds to the probabilities of transition from boundary nodes to internal nodes, R corresponds to the probabilities of transition from internal to boundary nodes, Q corresponds to the probabilities of transition between internal nodes, E is the absorption by boundary nodes.

For a Markov absorbing chain a fundamental matrix is [9]

$$N = (E - Q)^{-1} = \sum_{m=0}^{\infty} Q^m$$

(see (2)). Since the process is absorbed at boundary nodes with probability unity, $\lim_{m \to \infty} Q^m = 0$; see [9].

The elements of a fundamental matrix are $n_{ij} = M_i(u_j)$, where u_j is a function equal to the number of instants of occurrence of a process at internal node j; i.e., $n_{ij} = M_i(u_j)$ is the mathematical expectation of the number of visits to the node j by a process that leaves internal node i up to its absorption. If a Markov absorbing chain corresponds to a steady-state temperature field, a fundamental matrix of it corresponds to a fundamental solution of the linear equation of steady-state heat conduction (i.e., a discrete analog of it described by a system of equations in finite differences); we described above the physical meaning of a fundamental solution.

The elements b_{ik} of matrix (3) determine the probability that a process that leaves internal mode i will be absorbed boundary node k. When the net problem of steady-state heat conduction does not contain heat-release sources, knowing the influence coefficients from (3) will suffice to solve it. The elements of matrix (3) can be determined by a simpler method after solving a system of algebraic equations obtained from the condition [9]

$$PB^* = B^*,$$

where

$$B^* = \left|\begin{array}{c|c} E & 0 \\ \hline B & 0 \end{array}\right|$$

is a matrix of size $r \times r$.

The issues of exactness of the obtained results, convergence, stability, etc. are solved in the same way as in the finite-difference method; voluminous results obtained for this method [2] can be also used for our solutions.

Clear-cut computational procedures, stability and universality of solutions, use of the products of matrices as the basic mathematical operation, which requires time an order of magnitude less on specialized matrix processors than on ordinary computers, and advantageous use of the method of Green's function in classical problems of mathematical physics permit the hope that the above material will attract the attention of engineers and scientists to the development of similar methods for other problems of mathematical physics and their use in practical work.

NOTATION

P, R, Q, N, B, matrices; A^{-1}, inverse matrix of A; E, unit matrix; 0, zero matrix; T, T_c, T_k, T_i, temperature, ambient temperature, temperature at nodes k, i [K]; x, y, coordinates of the net; h, spatial step of the net [m]; λ, thermal conductivity [W/(m·K)]; q_v, q_s, internal heat sources [W/m^3, W/m^2], a_{id}, n_{ij}, b_{ik}, matrix elements; l, number of measurements ($l = 1, 2, 3$ for a one-, two-, and three-dimensional field, respectively); p, probability; M, mathematical expectation; α_k, heat-transfer coefficient at node k [W/(m^2·K)].

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